Ring Theory 9

31 March 2024 16:59

Rings of Fractions:-

R is a comm. ring. A subset $S \subset R$ is called multiplicative if $1 \in S$ and $\forall\, s, t \in S$ we have $st \in S$.

S is said to be proper if $0 \notin S$ and zero divisors $\notin S$, i.e., if $s \in S$, $r \in R$ and $sr = 0$ then $r = 0$ or zero divisor

Theorem:- Let R be a comm ring and $S \subset R$ be a multiplicative set and $I \subset R$ be an ideal. If $S \cap I = \phi$, then $\exists$ a prime ideal p containing I and $S \cap p = \phi$

Using Zorn's lemma we get p is maximal for $R \setminus S$

Proof:- Let $ab \in p$ and $a, b \notin p$

$(p + \langle a \rangle) \cap S \neq \phi$, $(p + \langle b \rangle) \cap S \neq \phi$

$(p + \langle a \rangle)(p + \langle b \rangle) \cap S \neq \phi$

$\Rightarrow (p^2 + \underbrace{p\langle a \rangle + p\langle b \rangle + \langle a \rangle \langle b \rangle}_{\in p}) \cap S \neq \phi$

$\Rightarrow \Leftarrow$

So we must have $a \in p$ or $b \in p$ $\Rightarrow$ p is prime ideal

R is a comm. ring and S is multiplicative set $S \subset R$

for $(a, s), (b, t) \in R \times S$ we define

$$(a, s) \sim (b, t) \iff (\exists\, s' \in S)(s'(at - bs) = 0)$$

If S is proper then $(a, s) \sim (b, t) \iff at - bs = 0$

$\sim$ is an equivalence relation.

$\sim$ is an equivalence relation.

$(a,s)\sim(b,t)$ and $(b,t)\sim(c,d)$

$\Rightarrow \exists\, s', s'' \in S$ such that $s'(at-bs)=0$ and $s''(bd-ct)=0$

$\Rightarrow s's''d(at-bs)=0$ and $s's''s(bd-ct)=0$

$\Rightarrow s's''(adt - cst) = 0$ and as $s's''t \in S$ we get,

$\Rightarrow s's''t(ad-cs)=0$

$(a,s)\sim(c,d)$

•> If $s,t\in S$ and $a\in R$ then $\frac{at}{st}=\frac{a}{s}$ as $1\cdot(ats-ast)=0$

•> If $s,t\in S$ then both $\frac{s}{t}$ and $\frac{t}{s} \in S^{-1}R$

•> If $0\in S$ then $S^{-1}R$ has a single element

$\frac{0}{1}=\frac{1}{0}$

$(a,s)=(0,1)$ $\quad 0(10-01)=0$

Proof:- $(1,0)\in S^{-1}R$

Let $(a,s)\in S^{-1}R$

We get, $(a,s)\sim(1,0)$ as $\exists\, 0\cdot(a\cdot 0 - s\cdot 1)=0$ $\Rightarrow S^{-1}R$ is singleton

So $(a,s)=(1,0)$ $\forall\, a,s\in R\times S$

Theorem:- Let R be a comm ring and $S\subset R$ be multiplicative set. Under addition and multiplication of fractions $S^{-1}R$ is a commutative ring, called the localization of R at S

$$\frac{a}{s}+\frac{b}{t}=\frac{at+bs}{st} \qquad \frac{a}{s}\cdot\frac{b}{t}=\frac{ab}{st}$$

•> $N^{-1}\mathbb{Z}$ is a field and is $\mathbb{Q}$

•> $R=\mathbb{Z}/6\mathbb{Z}$ and $S=\{1,2,2^2,2^3,\dots\}=\{1,2,4\}$.

$\frac{0}{1}=\frac{0}{2}=\frac{0}{4}=\frac{3}{1}=\frac{3}{2}$ $\qquad \frac{3}{1}=\frac{0}{2}$ $\qquad \frac{3}{2}=\frac{0}{4}$ $\qquad \frac{3}{4}=$

$\frac{1}{-}=\frac{2}{-}=\frac{4}{-}$

$$\frac{1}{1} = \frac{2}{2} = \frac{4}{4}$$

$$\frac{2}{1} = \frac{4}{2} = \frac{2}{4} \quad \text{and so on.}$$

The localization $S^{-1}R$ comes with a natural map

$$\varphi_S : R \to S^{-1}R$$
$$a \to \frac{a}{1}$$

For any $s \in S$, $\varphi(s) = \frac{s}{1}$ is a unit in $S^{-1}R$ as $\frac{1}{s} \in S^{-1}R$

$\Rightarrow \varphi(S) \subset (S^{-1}R)^{\times}$

<u>Lemma</u>:- Let R be a commutative ring with multiplicative set S. The natural map φ_S is an embedding iff S is proper.

Proof:- Suppose φ_S is an embedding $\Rightarrow \varphi_S$ is injective

Let $a \in R$, $s \in S$ such that $sa = 0$. Then

$\varphi(sa) = \varphi(s)\varphi(a)$. and $\varphi(s)$ is a unit. $\Rightarrow \varphi(a) = 0$

$\Rightarrow a = 0 \Rightarrow S$ is proper.

Suppose S is proper and let $\varphi_S(a) = \frac{a}{1} = \frac{0}{1} \Rightarrow a = 0$

$\varphi_S(a) = \varphi_S(b) \Rightarrow \frac{a}{1} = \frac{b}{1} \Rightarrow s'(a-b) = 0 \Rightarrow a = b$ as s' is not zero divisor

$\Rightarrow \varphi_S$ is embedding.

<u>Theorem (Universal Property of Localization)</u>:-

Let R be a comm ring and $S \subset R$ is a multiplicative set.

If $\psi : R \to R'$ is a ring homomorphism satisfying $\psi(R) \subset (R')^{\times}$, then ψ lifts to a unique homomorphism

$\hat{\psi} : S^{-1}R \to R'$

$S^{-1}R \quad \hat{\psi}$

$\hat{\psi} : S^{-1}R \to R'$

$$\begin{tikzcd} S^{-1}R \arrow[dr, "\hat{\psi}"] & \\ R \arrow[u, "\varphi"] \arrow[r, "\psi"'] & R' \end{tikzcd}$$

$\ker \hat{\psi} = S^{-1} \ker \psi$

If ψ is an embedding then so is $\hat{\psi}$.

<u>Example</u>:- $R = \mathbb{Z}/6\mathbb{Z}$ and $S = \{1, 2, 4\}$

Let $\pi : \mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z}$ be the natural surjection.

$\pi(2) = 2$ and $\pi(4) = 1$ are units of $\mathbb{Z}/3\mathbb{Z}$

$\hat{\pi} : S^{-1}R \to \mathbb{Z}/3\mathbb{Z}$

$\ker \hat{\pi} = S^{-1} \ker \pi = \frac{0}{s}, \frac{3}{s} = \frac{0}{s} = \frac{0}{1}$; $\frac{3}{s} = \frac{6}{2s} = \frac{0}{s}$; $\frac{0}{1} = \frac{0}{2} = \frac{0}{3}$

$\Rightarrow \ker \hat{\pi} = \left\{ \frac{0}{1} \right\}$

$\hat{\pi}$ is an embedding
$\hat{\pi}$ is also surjective as π is surjective
$\Rightarrow$ $\hat{\pi}$ is an isomorphism.

$\hat{\pi} : S^{-1}R \to \mathbb{Z}/3\mathbb{Z}$

$\Rightarrow S^{-1}R \cong \mathbb{Z}/3\mathbb{Z}$